TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 12, December 1995

e-INVARIANTS AND FINITE COVERS. II

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ABSTRACT. Let $\widetilde{M} \downarrow M$ be a finite covering of closed framed manifolds. By the Pontrijagin-Thom construction both $\widetilde{M}$ and M define elements in the stable homotopy ring of spheres π_*^s. Associated to $\widetilde{M}$ and M are their e-invariants $e_L(\widetilde{M})$, $e_L(M) \in \mathbb{Q}/\mathbb{Z}$. If $\widetilde{N} \downarrow N$ is a finite covering of closed oriented manifolds, then there is a related invariant $I_\Delta(\widetilde{N} \downarrow N) \in \mathbb{Q}$ of the diffeomorphism class of the covering. In a previous paper we examined the relation between these invariants. We reduced the determination of $e_L(\widetilde{M}) - pe_L(M)$, as well as $I_\Delta(\widetilde{N} \downarrow N)$, for a p-fold cover, to the evaluation of certain sums of roots of unity. In this sequel we show how the invariant theory of the cyclic group $\mathbb{Z}/p$ may be used to evaluate these rums. For example we obtain

$$\sum_{\substack{\zeta^p=1 \\ \zeta\neq 1}} \frac{(1+\zeta)(1+\zeta^{-1})}{(1-\zeta)(1-\zeta^{-1})} = \frac{(p-1)(p-2)}{3}$$

which may be used to determine the value of I_Δ in degrees congruent to 3 mod $2(p-1)$ for odd primes p.

1. BACKGROUND AND DESCRIPTION OF THE PROBLEM

Let $(\widetilde{M} \downarrow M, \Phi)$ be a finite covering of closed framed manifolds. The framing Φ of M induces a compatible framing $\widetilde{\Phi}$ of $\widetilde{M}$. The framed manifolds $(\widetilde{M}, \widetilde{\Phi})$ and (M, Φ) define elements $[\widetilde{M}, \widetilde{\Phi}]$ and $[M, \Phi]$ in the framed bordism ring [12] Ω_*^{fr}. By the Pontrjagin-Thom construction [12] Ω_*^{fr} is isomorphic to the stable homotopy ring of spheres π_*^s. Associated to $[\widetilde{M}, \widetilde{\Phi}]$ and $[M, \Phi]$ are their e-invariants [1], [8] $e_L(\widetilde{M})$, $e_L(M) \in \mathbb{Q}/\mathbb{Z}$. The invariant e_L is essentially the odd primary part of the complex e-invariant $e_{\mathbb{C}}$ introduced by J. F. Adams [1]. See [8] for the precise definition and relationships.

Contrary to expectation the invariant e_L (and also $e_{\mathbb{C}}$) do not behave multiplicatively, i.e. it is not in general true that

$$e_L(\widetilde{M}, \widetilde{\Phi}) = de_L(M, \Phi)$$

where d is the number of sheets of the covering. In [8] we studied the difference

$$e_\Delta(\widetilde{M} \downarrow M, \Phi) = e_L(\widetilde{M}, \widetilde{\Phi}) - de_L(M, \Phi),$$

which for regular p-fold coverings defines a homomorphism

$$e_\Delta : \Omega_*^{\mathrm{fr}}(B\mathbb{Z}/p) \cong \pi_*^s(B\mathbb{Z}/p) \to \mathbb{Q}/\mathbb{Z},$$

Received by the editors October 6, 1994 and, in revised form, February 21, 1995.
1991 *Mathematics Subject Classification*. Primary 13A50, 55P42, 55Q45.

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where $\Omega_*^{\mathrm{fr}}(\quad)$ denotes the framed bordism homology functor.

Let p be an odd prime. For a regular p-fold covering of closed oriented odd dimensional manifolds $\widetilde{N} \downarrow N$ there is the invariant $I_\Delta(\widetilde{N} \downarrow N) \in \mathbb{Q}$ defined as follows. Choose an integer k such that $k[\widetilde{N} \downarrow N] = 0 \in \Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p)$. This is always possible since in odd dimensions all classes are torsion [4]. Let $\widetilde{W} \downarrow W$ be a null bordism of $\widetilde{N} \downarrow N$ so that $\partial(\widetilde{W} \downarrow W) = k(\widetilde{N} \downarrow N)$ for some positive integer k and set

$$I_\Delta(\widetilde{N} \downarrow N) = \frac{-1}{k}(\mathrm{Index}(\widetilde{W}) - p \cdot \mathrm{Index}(W)).$$

The rational number so defined is independent of the choices made and defines a diffeomorphism invariant of $\widetilde{N} \downarrow N$ which vanishes if the dimension is not congruent to $-1 \bmod 4$ (see [8]). The residue class $I_\Delta(\widetilde{N} \downarrow N) \in \mathbb{Q}/\mathbb{Z}$ is an invariant of the bordism class $[\widetilde{N} \downarrow N] \in \Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p)$.

The invariants e_Δ and I_Δ are related by the commutative triangle

$$\begin{tikzcd}
\Omega_*^{\mathrm{fr}}(B\mathbb{Z}/p) \arrow[rr, "h"] \arrow[dr, "e_\Delta"'] & & \Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p) \arrow[dl, "I_\Delta"] \\
& \mathbb{Q}/\mathbb{Z} &
\end{tikzcd}$$

where h is the Hurewicz map, as was shown in [8, Theorem 5].

The structure of $\Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ was determined by Conner and Floyd [4] who described several sets of generators and relations for $\Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ as module over Ω_*^{SO}. To understand the behavior of I_Δ, and therefore e_Δ, it suffices to evaluate I_Δ on one of the standard sets of generators. We choose to do this by applying the Atiyah-Bott fixed point theorem to certain actions of $\mathbb{Z}/p$ on products of projective spaces. This was described in [8] and leads to the need to evaluate an expression of the form

$$(\star) \qquad \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \prod_{j=1}^{2m} \frac{(\zeta^j + 1)}{(\zeta^j - 1)},$$

where $\mathbb{Z}/p$ has been identified with the multiplicative group of complex pth roots of unity. The idea for evaluating expressions of this form is one I learned from the papers of L. Solomon [10] and R. P. Stanley [11]. In particular, in reworking Stanley's example [11, §5, Example 1] for [9, §4.2, Example 2] I realized that I had seen these types of numbers before. In writing up the theorem of Molien-Solomon for Chapter 9 of [9] I remembered where. The result is this manuscript, the sequel to [8].

The advent of computer algebra programs[1] makes the evaluation of formulas such as $(\star)$ feasible, and so perhaps the explicit formula

$$\sum_{\substack{\zeta^p = 1 \\ \zeta \neq 1}} \frac{(1+\zeta)(1+\zeta^{-1})}{(1-\zeta)(1-\zeta^{-1})} = \frac{(p-1)(p-2)}{3}$$

[1]The FORTRAN program I wrote in 1973 was not up to the task of evaluating the following formula for $p > 11$.

is not all that exciting. While this may be true, the method used here to obtain this result is of considerable value. It explains why, for example, Brieskorn [3] was able to obtain a combinatorial formula for the signature of the affine variety

$$V_a = \{(z_0, \dots, z_n) \mid z_0^{a_0} + \cdots + z_n^{a_n} = 1\}$$

whereas Zagier, [13, p. 232, formula (3)], obtains an analytic one. In both cases something is being counted, and the correct interpretation of what is being counted makes the equivalence of such formulas transparent. Thus the thrust of this note is not the result, but the method.

The Atiyah-Bott fixed point theorem yields expressions such as the one above when studying the *defect* of a $\mathbb{Z}/p$-action at a fixed point (see [7] and [13] and the references there).

Defect sums are defined by

$$\mathbf{def}(p\,; a_1, \dots, a_n) = \sum_{\substack{\zeta^p=1\\ \zeta\neq 1}} \frac{(\zeta^{a_1}+1)\cdots(\zeta^{a_n}+1)}{(\zeta^{a_1}-1)\cdots(\zeta^{a_n}-1)}$$

or equivalently

$$\mathbf{def}(p\,; a_1, \dots, a_n) = \sum_{k=1}^{p-1} \cot\left(\frac{\pi \mathbf{i} k a_1}{p}\right) \dots \cot\left(\frac{\pi \mathbf{i} k a_n}{p}\right)$$

where $\mathbf{i} = \sqrt{-1}$. In [7] and [13] Hirzebruch and Zagier analyze such cotangent sums and show that they may be evaluated by means of combinatorial formulas. From the point of view of invariant theory such combinatorial formulas become clear. Namely, one has two ways to compute the Poincaré series of the ring of invariants of a finite group. First one can use the theorem of Molien (or in our case Molien-Solomon) which for a cyclic group yields a defect-like sum. Secondly, again when the group is cyclic, the group action on the algebra of polynomials or differential forms sends monomials to monomials. Hence there is a set of congruences that must be fullfilled by the exponents of invariant monomials. Counting the number of solutions to such congruences gives an alternate, combinatorial formula, for the coefficients in the Poincaré series.

It is hoped that this note explains the invariant theoretic viewpoint for evaluating sums such as defect sums in an adequate manner. Section 2 describes the generators of $\Omega_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ that we find convenient. In the third section we describe how invariant theory leads to a combinatorial formula for

$$A_{2m}(p) = \sum_{\substack{\zeta\in\mathbb{Z}/p\\ \zeta\neq 1}} \prod_{i=1}^{m} \frac{(\zeta^i+1)(\zeta^{-i}+1)}{(\zeta^i-1)(\zeta^{-i}-1)}$$

(or any other defect-like sum) and in section 4 we evaluate this sum for $2m \equiv 3 \bmod 2(p-1)$. This is the only really new result here. However the methods used in this note are applicable to other problems arising from [7], [13], and [8] such as the determination of the index of the complex p-drics, that is of the complex manifold

$$\mathbf{Q}_n(p) = \{[z] \in \mathbb{CP}(n+1) | z_0^p + z_1^p + \cdots + z_{n+1}^p = 0\}$$

of real dimension $2n$ (see for example [5], [6], [7]).

2. Application of the Atiyah-Bott fixed point theorem to the problem

Let us begin by describing the generators of $\widetilde{\Omega}_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ that we find convenient for our purposes. (N. B. These differ slightly from the generators used in [8].) Let $\lambda = \exp(2\pi\mathrm{i}/p)$ and introduce the matrices $T_n \in \mathrm{GL}(n, \mathbb{C})$ by:

Case: $n = 2m$.

$$T_{2m} = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & \cdots & 0 & \\ 0 & \lambda^2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots & \\ 0 & \cdots & \cdots & \lambda^m & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda^{-m} & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{GL}(2m, \mathbb{C}).$$

Case: $n = 2m+1$.

$$T_{2m+1} = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda^2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & \lambda^m & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda^{m+1} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda^{-m} & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda^{-1} \end{bmatrix} \in \mathrm{GL}(2m+1, \mathbb{C}).$$

The corresponding representation of $\mathbb{Z}/p$ will be denoted by

$$\sigma_n \colon \mathbb{Z}/p \hookrightarrow \mathrm{GL}(n, \mathbb{C}).$$

For $n = a(p-1)$ the representation σ_n agrees (up to permutation of factors) with the representation $a\tilde{p}$ used in [8]. If we denote by $S(\sigma_n) \downarrow L(\sigma_n)$ the p-fold cover corresponding to σ_n of the unit sphere in $\mathbb{C}^n$ over the generalized lens space then we obtain elements

$$[S(\sigma_n) \downarrow L(\sigma_n)] \in \Omega_{2n-1}^{\mathrm{SO}}(B\mathbb{Z}/p).$$

As n ranges over $\mathbb{N}$ these provide a minimal set of generators of $\widetilde{\Omega}_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ as Ω_*^{SO}-module. The relation between these generators and the other sets of standard generators for $\widetilde{\Omega}_*^{\mathrm{SO}}(B\mathbb{Z}/p)$ (such as those used in [8]) is well understood in terms of formal group laws.

From [8, Proposition 11] and the Atiyah-Bott fixed point formula [2], [7] (see [8] the discussion following Proposition 12) one has the formula

$$I_\Delta(S(a\tilde{p} \oplus \sigma_b)) \downarrow L(\sigma(a\tilde{p} \oplus \sigma_b)) = \begin{cases} \dfrac{A_b(p)}{p^a} & \text{for } b \text{ even with } 1 \le b < p-1, \\ 0 & \text{for } b \text{ odd}, \end{cases}$$

where for even b, $b = 2m$,

$$A_{2m}(p) = \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \prod_{i=1}^{m} \frac{(\zeta^i + 1)(\zeta^{-i} + 1)}{(\zeta^i - 1)(\zeta^{-i} - 1)},$$

the sum extending over the pth roots of unity different from 1. The differences with the formula in [8] arise from the different choices of generators in [8] and the present note. The numbers $A_{2m+1}(p)$ are set equal to zero.

For $m = 1$ this reduces to

$$A_2(p) = \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \frac{(\zeta + 1)(\zeta^{-1} + 1)}{(\zeta - 1)(\zeta^{-1} - 1)}.$$

In the next section we will show how invariant theory may be used to convert the evaluation of $A_{2m}(p)$ into a combinatorial problem (counting solutions of linear equations in nonnegative integers). For $m = 1$ we solve the counting problem in section 3 obtaining the formula

$$A_2(p) = \frac{(p-1)(p-2)}{3}$$

and a recursion then leads to the formula

$$A_{a(p-1)+2} = \frac{(p-1)(p-2)}{3p^a}.$$

This determines I_Δ on the generators $S(\sigma_n) \downarrow L(\sigma_n)$, when $n = a(p-1) + 2$.

3. Reduction to invariant theory

Identify $\mathbb{Z}/p$ with the multiplicative group of complex pth roots of unity. We are concerned with the evaluation of the sum

$$A_{2m}(p) = \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \prod_{i=1}^{m} \frac{(\zeta^i + 1)(\zeta^{-i} + 1)}{(\zeta^i - 1)(\zeta^{-i} - 1)}.$$

Note that the sum extends over the complex roots of unity different from 1. The following discussion applies to other examples of this type and Solomon [10] provides a general framework for dealing with such computations.

The matrix $T_{2m} \in \mathrm{GL}(2m, \mathbb{C})$ defines a representation of $\mathbb{Z}/p$ on $\mathbb{C}^{2m}$ and hence also on

$$\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m] \otimes E[dx_1, \dots, dx_m, dy_1, \dots, dy_m],$$

which we may think of as the algebra of differential forms on $\mathbb{C}^{2m}$. (As a general reference on invariant theory we refer to [9].) We bigrade this algebra by assigning the polynomial generators (i.e., the xs and the ys) the bidegree $(1, 0)$ and their exterior derivatives (i.e., the dxs and the dys) the bidegree $(0, 1)$. The subalgebra of $\mathbb{Z}/p$-invariant differential forms

$$(\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m] \otimes E[dx_1, \dots, dx_m, dy_1, \dots, dy_m])^{\mathbb{Z}/p}$$

may be assigned as Poincaré series the double series $P(t, s)$ defined by

$$\sum_{i,j=0}^{\infty} \dim_{\mathbb{C}}((\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m] \otimes E[dx_1, \dots, dx_m, dy_1, \dots, dy_m])^{\mathbb{Z}/p}_{i,j}) t^i s^j$$

(see [9, §9.2]). The starting point for analyzing this series is the theorem of Molien-Solomon ([9, Theorem 9.2.2]) that computes this series in terms of character theory. Namely

$$P(t, s) = \frac{1}{p} \sum_{g \in \mathbb{Z}/p} \frac{\det(1 + g^{-1}s)}{\det(1 - g^{-1}t)}$$
$$= \frac{1}{p} \left\{ \sum_{\zeta \in \mathbb{Z}/p} \prod_{i=1}^{m} \frac{(1 + \zeta^i s)(1 + \zeta^{-i} s)}{(1 - \zeta^i t)(1 - \zeta^{-i} t)} \right\}.$$

Note that the sum runs over all the elements of $\mathbb{Z}/p$, not just those different from 1. Introduce the series

$$A_{2m}(p)(t, s) = \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \prod_{i=1}^{m} \frac{(1 + \zeta^i s)(1 + \zeta^{-i} s)}{(1 - \zeta^i t)(1 - \zeta^{-i} t)}.$$

Note that with these notations

$$A_{2m}(p)(t, s) = p \cdot P(t, s) - \left(\frac{1+s}{1-t}\right)^{2m}.$$

We apply a standard combinatorial paradigm to evaluate $A_{2m}(p)$. Namely, we evaluate the Poincaré series of

$$(\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m] \otimes E[dx_1, \dots, dx_m, dy_1, \dots, dy_m])^{\mathbb{Z}/p}$$

in another way, equate that with the formula of Molien-Solomon, solve for $A_{2m}(p)(t, s)$ and evaluate the resulting expression at $(t, s) = (1, 1)$.

The essential insight into our problem is that this is possible. The reason is because the action of $\mathbb{Z}/p$ on

$$\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_m] \otimes E[dx_1, \dots, dx_m, dy_1, \dots, dy_m]$$

sends monomials to monomials, and a monomial

$$x_1^{e_1} \dots x_m^{e_m} y_1^{d_1} \dots y_m^{d_m} dx_1^{\varepsilon_1} \dots dx_m^{\varepsilon_m} dy_1^{\delta_m} \dots d_m^{\delta_m}$$

is invariant if and only if

$$(*) \qquad (e_1 - d_1) + \cdots + (e_m - d_m) + (\varepsilon_1 - \delta_1) + \cdots + (\varepsilon_m - \delta_m) \equiv 0 \mod p$$

where $e_1, \dots, e_m, d_1, \dots, d_m$ are nonnegative integers and $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_m$ are 0 or 1. The bidegree of the preceding monomial is

$$(e_1 + \cdots + e_m + \varepsilon_1 + \cdots + \varepsilon_m, d_1 + \cdots + d_m + \delta_1 + \cdots + \delta_m),$$

so if we denote by $a(i, j)$ the number of solutions to $(*)$ of bidegree (i, j) we have formally

$$P(t, s) = \sum_{i, j}^{\infty} a(i, j) t^i s^j.$$

This should be compared to the discussion [7, §10] of the index (or signature) of the complex p-dric $\mathbf{Q}_{2n}(p)$. Hirzebruch and Zagier apply number theoretic considerations to obtain a *combinatorial formula* for $\operatorname{Index}(\mathbf{Q}_{2n}(p))$. These formulae[2] may be recovered from [5, Proposition 4.3] and the above considerations.

A further aid in the computation results from the theorem of Eagon-Hochster (see[3] [9, Theorem 7.7]) which assures us that the ring of invariants is Cohen-Macaulay. Since $x_1^p, \ldots, x_m^p, y_1^p, \ldots, y_m^p$ is visibly a system of parameters for the algebra of invariants we may write

$$(\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m] \otimes E[dx_1, \ldots, dx_m, dy_1, \ldots, dy_m])^{\mathbb{Z}/p}$$

as a free finitely generated module over $\mathbb{C}[x_1^p, \ldots, x_m^p, y_1^p, \ldots, y_m^p]$ the subalgebra on the finite number of generators

$$x_1^{e_1} \ldots x_m^{e_m} y_1^{d_1} \ldots y_m^{d_m} dx_1^{\varepsilon_1} \ldots dx_m^{\varepsilon_m} dy_1^{\delta_m} \ldots d_m^{\delta_m}$$

where, since the monomial is invariant, the exponents satisfy $(*)$ and

$$(**) \qquad 0 \le e_i, d_j \le p-1, \qquad i, j = 1, \ldots, m.$$

Therefore we may conclude

$$P(t, s) = \frac{Q(t, s)}{(1-t^p)^{2m}}$$

where $Q(t, s)$ is the polynomial defined by

$$Q(t, s) = \sum_{j=0}^{m} \sum_{i=0}^{2m(p-1)} b(i, j) t^i s^j$$

and $b(i, j)$ is the number of solutions to $(*)$ and $(**)$ of bidegree (i, j). Already this formal analysis yields the (for me remarkable) fact

$$\begin{aligned} A_{2m}(p)(t, s) &= \sum_{\substack{\zeta \in \mathbb{Z}/p \\ \zeta \neq 1}} \prod_{i=1}^{m} \frac{(1+\zeta^i s)(1+\zeta^{-i} s)}{(1-\zeta^i t)(1-\zeta^{-i} t)} \\ &= p\left[\frac{Q(t, s)}{(1-t^p)^{2m}}\right] - \frac{(1+s)^{2m}}{(1-t)^{2m}} \end{aligned}$$

and hence that

$$A_{2m}(p) = \left.\frac{p \cdot Q(t, s) - (1+s)^{2m}(1+t+\cdots+t^{p-1})^{2m}}{(1-t^p)^{2m}}\right|_{\substack{s=1 \\ t=1}}$$

and since s does not appear in the denominator

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[2] If the formula in [5, 4.3] is rewritten

$$\operatorname{Index}(\mathbf{Q}_{2n}(q)) = 1 + \frac{1}{p} \sum_{\zeta^q = -1} \left(\frac{\zeta+1}{\zeta-1}\right)^{2n+2},$$

a connection with the invariant theory of the group of $2p$th roots of unity becomes apparent.

[3] In [9, Theorem 7.7] the result is stated for polynomial invariants. The extension to the invariants in the algebra of differential forms is straightforward. In fact the proof of [9, Theorem 7.7] works with no essential changes.

$$A_{2m}(p) = \left.\frac{p \cdot Q(t, 1) - (2)^{2m}(1 + t + \cdots + t^{p-1})^{2m}}{(1 - t^p)^{2m}}\right|_{t=1}.$$

(I wish I had known this in 1973.)

4. Evaluation of $A_{a(p-1)+2}$ mod p

For a positive integer n and an odd prime p write $n = a(p-1) + b$ where $0 \le b < p - 1$. The representation σ_n is equivalent to $a\tilde{\rho} \oplus \sigma_b$. Observe for any pth root of unity $\zeta \ne 1$ that

$$\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\} = \{\lambda, \lambda^2, \ldots, \lambda^{p-1}\}$$

where $\lambda = \exp(2\pi \mathbf{i}/p)$ and hence

$$\begin{aligned}\frac{(1+\zeta s)(1+\zeta^2 s)\cdots(1+\zeta^{p-1}s)}{(1-\zeta t)(1-\zeta^2 t)\cdots(1-\zeta^{p-1}t)} &= \frac{(1+\lambda s)(1+\lambda^2 s)\cdots(1+\lambda^{p-1}s)}{(1-\lambda t)(1-\lambda^2 t)\cdots(1-\lambda^{p-1}t)}\\ &= \frac{1 - s + s^2 - \cdots + s^{p-1}}{1 + t + t^2 + \cdots + t^{p-1}}.\end{aligned}$$

From this equality we obtain a recursion formula for the Poincaré series of the rings of invariants $(\mathbb{F}[V] \otimes E[V])^{\mathbb{Z}/p}$ as a function of n. For this purpose it is convenient to introduce a definition.

Definition. Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group G. The **reduced Poincaré series** of the ring of invariants $(\mathbb{F}[V] \otimes E[V])^G$ is the double series

$$\widetilde{P}((\mathbb{F}[V] \otimes E[V])^G, t, s) = P((\mathbb{F}[V] \otimes E[V])^G, t, s) - \frac{1}{|G|}\left[\frac{1+s}{1-t}\right]^n.$$

In this notation we have

$$A_{2m}(p) = p \cdot \widetilde{P}(\mathbb{F}[z_1 \ldots z_n] \otimes E[dz_1, \ldots, dz_n]^{\mathbb{Z}/p}, t, s)\Big|_{\substack{s=1\\ t=1.}}$$

By the theorem of Molien-Solomon [9, §9.2] we have the formula

$$\widetilde{P}((\mathbb{F}[V] \otimes E[V])^G, t, s) = \frac{1}{|G|} \sum_{\substack{g \in G\\ g \ne 1}} \frac{\det(1 + g^{-1}s)}{\det(1 - g^{-1}t)}.$$

Proposition 4.1. *Let n be a positive integer and p an odd prime, and write $n = a(p-1) + b$ with $0 \le b \le p-1$. Then*

$$\begin{aligned}&\widetilde{P}((\mathbb{F}[z_1, \ldots, z_n] \otimes E[dz_1, \ldots, dz_n])^{\mathbb{Z}/p}, t, s)\\ &\quad = \left[\frac{1 - s + s^2 - \cdots + s^{p-1}}{1 + t + t^2 + \cdots + t^{p-1}}\right]^a \widetilde{P}((\mathbb{F}[z_1, \ldots, z_b] \otimes E[dz_1, \ldots, dz_b])^{\mathbb{Z}/p}, t, s).\end{aligned}$$

Proof. By the theorem of Molien-Solomon (loc. cit.) we have

$$\begin{aligned}
&\widetilde{P}((\mathbb{F}[z_1, \dots, z_n] \otimes E[dz_1, \dots, dz_n])^{\mathbb{Z}/p}, t, s) \\
&\quad = \frac{1}{p} \sum_{1 \neq \zeta \in \mathbb{Z}/p} \prod_{i=1}^{n} \left(\frac{1 + \zeta^i s}{1 - \zeta^i t} \right) \\
&\quad = \frac{1}{p} \sum_{1 \neq \zeta \in \mathbb{Z}/p} \left[\prod_{i=1}^{p-1} \left(\frac{1 + \zeta^i s}{1 - \zeta^i t} \right) \right]^a \prod_{i=1}^{b} \left(\frac{1 + \zeta^i s}{1 - \zeta^i t} \right) \\
&\quad = \left[\frac{1 - s + s^2 - \cdots + s^{p-1}}{1 + t + t^2 + \cdots + t^{p-1}} \right]^a \frac{1}{p} \sum_{1 \neq \zeta \in \mathbb{Z}/p} \prod_{i=1}^{b} \left(\frac{1 + \zeta^i s}{1 - \zeta^i t} \right) \\
&\quad = \left[\frac{1 - s + s^2 - \cdots + s^{p-1}}{1 + t + t^2 + \cdots + t^{p-1}} \right]^a \widetilde{P}((\mathbb{F}[z_1, \dots, z_b] \otimes E[dz_1, \dots, dz_b])^{\mathbb{Z}/p}, t, s)
\end{aligned}$$

as claimed. □

Since

$$A_n(p) = p \cdot \widetilde{P}((\mathbb{F}[V] \otimes E[V])^G, 1, 1)$$

we obtain an algebraic proof of [8, Proposition 11].

Corollary 4.2. *Let n be a positive integer and p an odd prime, and write $n = a(p-1) + b$ with $0 \leq b \leq p-1$. Then*

$$A_n(p) = \frac{1}{p^a} A_b(p).$$

Proof. We have

$$\begin{aligned}
A_n(p) &= p \cdot \widetilde{P}((\mathbb{F}[z_1, \dots, z_n] \otimes E[dz_1, \dots, dz_n])^{\mathbb{Z}/p}, t, s)|_{\substack{s=1 \\ t=1}}, \\
&= \frac{1}{p^a} p \cdot \widetilde{P}((\mathbb{F}[z_1, \dots, z_b] \otimes E[dz_1, \dots, dz_b])^{\mathbb{Z}/p}, t, s)|_{\substack{s=1 \\ t=1}}
\end{aligned}$$

as required. □

Thus to compute $A_n(p)$ it is sufficient to compute $A_b(p)$ for $b = 0, 1, \dots, p-1$. We turn to the evaluation of $A_2(p)$. The representation σ_2 of $\mathbb{Z}/p$ is given by the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

From the theorem of Molien-Solomon [9, Theorem 9.2.1] the Poincaré series of

$$(\mathbb{C}[x, y] \otimes E[dx, dy])^{\mathbb{Z}/p}$$

is given by the double series

$$P(t, s) = \frac{1}{p} \left\{ \sum_{\zeta \in \mathbb{Z}/p} \frac{(1 + \zeta s)(1 + \zeta^{-1} s)}{(1 - \zeta t)(1 - \zeta^{-1} t)} \right\}.$$

The elements x^p, y^p are a homogeneous system of parameters so

$$(\mathbb{C}[x, y] \otimes E[dx, dy])^{\mathbb{Z}/p}$$

is a free finitely generated module over $\mathbb{C}[x^p, y^p]$ with basis the classes

$$x^e y^d dx^\varepsilon dy^\delta$$

satisfying

$$e - d + \varepsilon - \delta \equiv 0 \mod p, \qquad 0 \le e, d \le p-1, \quad \varepsilon, \delta = 0, 1.$$

Note that

$$p \ge e + \varepsilon \ge e - d + \varepsilon - \delta \ge -d - \varepsilon \ge -p$$

so there are two extreme cases given by

$$e = p-1, \varepsilon = 0, \quad d = 0 = \delta,$$
$$e = 0 = \varepsilon, \quad d = p-1, \delta = 1$$

corresponding to the invariants $x^{p-1}dx$, $y^{p-1}dy$. In all other cases the basis elements satisfy

$$e - d + \varepsilon - \delta = 0, \qquad 0 \le e, d \le p-1, \quad \varepsilon, \delta = 0, 1.$$

To list all the generators we divide into three cases:

Case: $\varepsilon = \delta$. Then $e = d$ and we obtain as invariants

$$1, xy, (xy)^2, \dots, (xy)^{p-1} : \varepsilon = 0 = \delta,$$
$$dx\,dy, xy\,dx\,dy, \dots, (xy)^{p-1} dx\,dy : \varepsilon = 1 = \delta.$$

Case: $\varepsilon = 1, \delta = 0$. In this case we have $e-d+1 = 0$ or equivalently $e = d-1$ which gives as invariants

$$y\,dx, xy^2\,dx, x^2y^3\,dx, \dots, x^{p-2}y^{p-1}\,dx.$$

Case: $\varepsilon = 0, \delta = 1$. This case is fully symmetric to the preceding one giving as invariants the elements

$$x\,dy, x^2y\,dy, x^3y^2\,dy, \dots, x^{p-1}y^{p-2}\,dy.$$

So a complete list of the basis elements for $(\mathbb{C}[x, y] \otimes E[dx, dy])^{\mathbb{Z}/p}$ as a $\mathbb{C}[x^p, y^p]$ module is

$$1, xy, (xy)^2, \dots, (xy)^{p-1},$$
$$dx\,dy, xy\,dx\,dy, (xy)^2\,dx\,dy, \dots, (xy)^{p-1}\,dx\,dy,$$
$$x\,dy, x^2y\,dy, \dots, x^{p-1}y^{p-2}\,dy,$$
$$y\,dx, xy^2\,dx, \dots, x^{p-2}y^{p-1}\,dx,$$
$$x^{p-1}\,dx, y^{p-1}\,dx.$$

Hence the Poincaré series of $(\mathbb{C}[x, y] \otimes E[dx, dy])^{\mathbb{Z}/p}$ is given by

$$P(t, s) = \frac{(1+s^2)(1+t^2+\cdots+t^{2(p-1)}+2s(t+t^3+\cdots+t^{2p-3})+2t^{p-1}s}{(1-t^p)^2}$$

and so equating with the formula given by the theorem of Molien-Solomon we obtain

$$\left\{\sum_{\zeta\in\mathbb{Z}/p} \frac{(1+\zeta s)(1+\zeta^{-1}s)}{(1-\zeta t)(1-\zeta^{-1}t)}\right\}$$
$$= \frac{(1+s^2)(1+t^2+\cdots+t^{2(p-1)}+2s(t+t^3+\cdots+t^{2p-3})+2t^{p-1}s}{(1-t^p)^2}$$

and hence

$$A_2(p) = \sum_{\substack{\zeta^p=1\\ \zeta\neq 1}} \frac{(1+\zeta)(1+\zeta^{-1})}{(1-\zeta)(1-\zeta^{-1})} = \sum_{\substack{\zeta^p=1\\ \zeta\neq 1}} \frac{(1+\zeta s)(1+\zeta^{-1}s)}{(1-\zeta t)(1-\zeta^{-1}t)}\Bigg|_{\substack{s=1\\ t=1}} = p\cdot \widetilde{P}(t,s)\Bigg|_{\substack{s=1\\ t=1}}$$

$$= \left\{ p\left[\frac{(1+s^2)(1+t^2+\cdots+t^{2(p-1)}+2s(t+t^3+\cdots+t^{2p-3})+2t^{p-1}s}{(1-t^p)^2}\right] - \frac{(1+s)^2}{(1-t)^2}\right\}\Bigg|_{\substack{s=1\\ t=1}}$$

(put $s=1$)

$$= \left\{ p\left[\frac{2(1+t^2+\cdots+t^{2(p-1)}+2(t+t^3+\cdots+t^{2p-3})+2t^{p-1}}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ p\left[\frac{2(1+t+t^2+\cdots+t^{2(p-1)}+2t^{p-1}}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ p\left[\frac{2(1+t+t^2+\cdots+t^{(p-1)}+2(t^p+\cdots+t^{2p-3})+2t^{p-1}}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

(add and subtract $2t^{2p-1}$)

$$= \left\{ p\left[\frac{2(1+t+\cdots+t^{(p-1)})+2(t^p+\cdots+t^{2p-1})-2t^{2p-1}+2t^{p-1}}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ p\left[\frac{2(1+t+\cdots+t^{(p-1)})+2t^p(1+\cdots+t^{p-1})+2t^{p-1}(t^p-1)}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ p\left[\frac{2(1+t^p)(1+t+\cdots+t^{(p-1)}-2t^{p-1}(t^p-1)}{(1-t^p)^2}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

(we may factorize $(t^p-1)=(t-1)(t^{p-1}+\cdots+t+1)$ so we have a cyclotomic polynomial as common factor in numerator and denominator on the left)

$$= \left\{ p\left[\frac{2(1+t^p)-2t^{p-1}(t-1)}{(1-t^p)(1-t)}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ p\left[\frac{2+2t^p-2t^p+2t^{p-1}}{(1-t^p)(1-t)}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \left\{ 2p\left[\frac{t^{p-1}+1}{(1-t^p)(1-t)}\right] - \frac{4}{(1-t)^2}\right\}\Bigg|_{t=1}$$

$$= \frac{2p(t^{p-1}+1)-4(1+t+t^2+\cdots+t^{p-1})}{(1-t^p)(1-t)}\Bigg|_{t=1}.$$

Factor 2 out of the numerator and a cyclotomic out of the denominator, and evaluate the cyclotomic at $t=1$ to obtain

$$= \frac{2}{p}\left[\frac{p(t^{p-1}+1)-2(1+t+\cdots+t^{p-1})}{(1-t)^2}\right]\Bigg|_{t=1}.$$

The numerator has $(t-1)^2$ as a factor (the evaluation must be finite) so we may apply l'Hôpital's rule twice to perform the evaluation. We find for the second derivative of the numerator:

$$\begin{aligned}\frac{d^2}{dt^2}&(p(t^{p-1}+1)-2(1+t+\cdots+t^{p-1})|_{t=1}\\ &=[p(p-1)(p-2)t^{p-3}-2(2\cdot 1+3\cdot 2\cdot t+\cdots+(p-1)(p-2))t^{p-3}]|_{t=1}\\ &=p(p-1)(p-2)-2(2\cdot 1+3\cdot 2+\cdots+(p-1)(p-2).\end{aligned}$$

To simplify this expression note

$$j(j+1)=\frac{(j+1)(j)(j-1)!\cdot 2!}{(j-1)!\cdot 2!}=2\binom{j+1}{2}.$$

By standard recursion formulae for binomial coefficients and l'Hôpital's rule applied twice we obtain

$$\begin{aligned}A_2(p)&=\frac{2}{p}\left[\frac{p(t^{p-1}+1)-2(1+t+\cdots+t^{p-1})}{(1-t)^2}\right]\Bigg|_{t=1}\\ &=\frac{2}{p}\frac{1}{2}\left[p(p-1)(p-2)-\frac{2p(p-1)(p-2)}{3}\right]\\ &=\frac{(p-1)(p-2)}{3}.\end{aligned}$$

Hence combining the preceding computation and 4.2 we obtain:[4]

Proposition 4.3. *Let n be a positive integer and p an odd prime, with $n \equiv 2 \pmod{p-1}$. Write $n=a(p-1)+2$. Then*

$$A_n(p)=\frac{(p-1)(p-2)}{3p^a}. \quad \square$$

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[4]I would like to thank Nelson Killius and Brian Wills for straightening out a calculational error in an earlier version of this result.

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